Math 4200
Monday September 14

1.5 harmonic functions, harmonic conjugates; 1.6 analtyic functions constructed via e^z and $\log z$. (We'll have more section 1.6 discussions on Wednesday.)

Announcements: HW is now due Friday (this week). ← I'll maybe change general due dates for HW to Fridays @ 11:59 pm
Quiz on Wed

On Friday's notes, showed $(e^z)' = e^z$

Review example Let $f(z) = \log z = \ln|z| + i\, \underbrace{arg}(z)$. Prove $f(z)$ is analytic with $\boxed{f'(z) = \frac{1}{z}}$, away from $z = 0$ (for any continuous branch choice i.e. by specifying $arg(z)$ continuously in a neighborhood of z). Do this three ways! Each of these is easier than trying to verify the limit definition directly.

~~didn't really use this~~ actually we did — to know that $f^{-1}(z)$ is analytic.

1) Inverse function theorem and chain rule.

recheck $e^{\log z} = z$? $k \in \mathbb{Z}$, different possibilities for arg.

$e^{\ln|z| + i \arg z + i(2\pi k)} \overset{?}{=} z$

$\arg z = \arg(x+iy)$
e.g. $\arctan(y/x)$ in 1^{st} & 4^{th} quad

$\underbrace{\underbrace{e^{\ln|z|}}_{|z|} e^{i \arg z}}_{\substack{z \\ \text{in polar form}}} \underbrace{e^{i(2\pi k)}}_{1} = z$ ✓.

Chain rule

$\underbrace{\frac{d}{dz} e^{\log z}}_{e^{\log z} \frac{d}{dz} \log z} = \frac{d}{dz} z = 1$

$\underbrace{e^{\log z}}_{z} \frac{d}{dz} \log z = 1 \Rightarrow \frac{d}{dz} \log z = \frac{1}{z}$ ✓

2) Rectangular Cauchy-Riemann equations plus continuous partials, via the Cauchy-Riemann Theorem. (how we showed $(e^z)' = e^z$).

$$\log(x+iy) = \underbrace{\underbrace{\ln|x+iy|}_{\frac{1}{2}\ln(x^2+y^2)}}_{u(x,y)} + i\, \underbrace{\underbrace{\arg(x+iy)}_{\text{e.g. } \arctan y/x}}_{v(x,y)} \quad (+2\pi k i)$$

Check. $u_x = v_y$ ...
$u_y = -v_x$

painful.

3) Polar coordinate CR equations, plus C^1. (You worked out the CR equations in polar coordinates in your last homework probably using 3220 chain rule; we can recover them quickly from the chain rule for curves, writing $f(z) = f\left(r e^{i\theta}\right)$.

$\log z = \ln|z| + i \arg z$
$\log(re^{i\theta}) = \ln r + i\theta$

$$\frac{\partial}{\partial r} f(re^{i\theta}) \overset{\text{CRFC}}{=} f'(re^{i\theta})\, e^{i\theta} \leftarrow \frac{\partial}{\partial r} re^{i\theta}$$

$$\frac{\partial}{\partial \theta} f(re^{i\theta}) = f'(re^{i\theta})\, ri e^{i\theta} \qquad \frac{\partial}{\partial \theta}(re^{i\theta})$$

$$\Rightarrow \frac{\partial f}{\partial \theta} = ri \frac{\partial f}{\partial r}$$

$$f = u + iv \Rightarrow u_\theta + i v_\theta = ri(u_r + i v_r) \overset{\text{Re, Im}}{\Longrightarrow} \boxed{\begin{cases} u_\theta = -r v_r \\ v_\theta = r u_r \end{cases}}$$

$\log(re^{i\theta}) = \underbrace{\ln r}_{u} + i \underbrace{\theta}_{v}$

$u_\theta = 0, \quad -rv_r = 0$ ✓

$v_\theta = 1, \quad ru_r = r\frac{1}{r} = 1$ ✓

1.5 Harmonic functions and harmonic conjugates.

Let $f(z) = f(x + i\,y) = u(x, y) + i\,v(x, y)$ be analytic in an open domain A, and assume u, v have continuous first and second partial derivatives. (The shorthand for this is $u, v \in C^2(A)$.) Then from Cauchy Riemann

$$\left[\begin{array}{l} u_x = v_y \Rightarrow u_{xx} = v_{yx} \\ u_y = -v_x \Rightarrow u_{yy} = -v_{xy} \end{array}\right.$$

we compute

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0.$$

(Recall from 3220 or multivariable calculus that $v_{yx} = v_{xy}$ when all second partial derivatives are continuous.)

<u>Def</u> Let $U(x, y)$ be a C^2 function in a domain $A \subseteq \mathbb{R}^2$. Then U is *harmonic* in A if it satisfies the partial differential equation

"Laplacian of u"

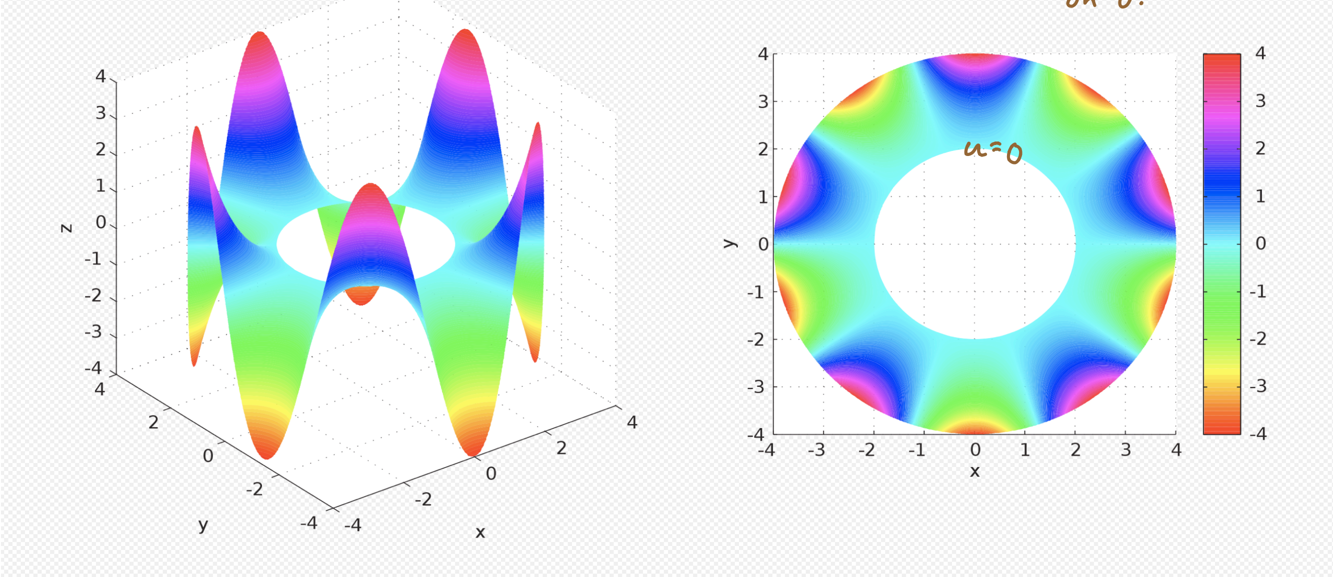
$$\nabla^2 u = \Delta u := U_{xx} + U_{yy} = 0.$$

<u>Def</u> The partial differential equation above is called *Laplace's equation*.

Harmonic functions are important in pure and applied math, as well as in physics. Also harmonic functions of three or more variables. If you've taken any class on partial differential equations or electro-magnetism, you've seen harmonic functions before. Here's the graph of a certain harmonic function defined on an annulus, taken from the Wikipedia page on harmonic functions. It could represent a the equilibrium temperature distribution on a thin metal plate, where the temperature values are specified as indicated on the inner and outer circles of the annulus.

$\Delta u(x,y) = 0$

unique equil. temp distrib's $u(x,y)$ once you specify fixed temps on ∂.



f analytic $\Rightarrow$ Ref harmonic (Im f is too).

converse...

Def Let $A \subseteq \mathbb{C}$ open, and let $u \in C^2(A)$ be a harmonic function. A function $v(x, y)$ such that

$$f(z) = f(x + i\,y) = u(x, y) + i\,v(x, y)$$

is analytic in A is called a *harmonic conjugate* to $u(x, y)$.

Theorem If $u(x, y) \in C^2(A)$ where A is an open *simply connected* domain. (A domain is called simply connected if it's connected and "has no holes". We'll discuss this concept more carefully in the next chapter.) Then there exists a harmonic conjugate $v(x, y)$ to $u(x, y)$, unique up to an additive constant.

proof: $u \in C^2(A)$, $u_{xx} + u_{yy} = 0$ is given. The system for finding $v(x, y)$ has to be consistent with the Cauchy-Riemann equations for f:

$$\begin{cases} v_x = P(x, y) & (= -u_y) \\ v_y = Q(x, y) & (= u_x) \end{cases} \qquad (P, Q) = (-u_y, u_x)$$

when is a vector field a gradient field.

the curl.

When you studied *conservative vector fields* and *Green's Theorem* in multivariable calculus you learned that a vector field $[P, Q]^T$ is actually the gradient of a function $v(x, y)$ locally if and only if the necessary condition that v_{xy} would equal v_{yx} holds:

- $$P_y = Q_x$$

In our case, since P, Q are partials of $u(x, y)$ this integrability condition reads as

$$-u_{yy} = u_{xx}$$

which holds since u is harmonic!

Example Let $u(x, y) = x\,y$. Show u is harmonic. (1) Then find its harmonic conjugate (2) $v(x, y)$ and identify the analytic function $f(z) = u(x, y) + i\,v(x, y)$. (3)

(1) $u_{xx} + u_{yy} = 0 + 0 = 0$ ✓

$u(x,y) = xy$

$u_x = y \qquad u_y = x$

$u_{xx} = 0 \qquad u_{yy} = 0$

$v(x,y) = -\frac{x^2}{2} + C(y) = \frac{1}{2}y^2 + D(x)$

$\Rightarrow \boxed{v(x,y) = -\frac{x^2}{2} + \frac{1}{2}y^2 + C}$

(2) $u + iv$ s.t. sats CR.

$$\text{CR}\begin{cases} v_x = -u_y \\ v_y = u_x \end{cases} \Rightarrow \begin{cases} v_x = -x \\ v_y = y \end{cases} \Longrightarrow \begin{aligned} v(x,y) &= \int \overbrace{v_x(x,y)}^{-x}\,dx = -\frac{x^2}{2} + C(y) \\ v(x,y) &= \int \underbrace{v_y(x,y)}_{y}\,dy = \frac{1}{2}y^2 + D(x) \end{aligned}$$

(3) $f(z) = xy + i\left(-\frac{x^2}{2} + \frac{y^2}{2}\right) + iC$

$z^2 = (x+iy)^2 = (x^2-y^2) + 2i\,xy$

$f(z) = \frac{1}{2i} z^2$!

Rigorous proof of key point for harmonic conjugate construction

(foreshadow Chapter 2)
Chptr 1: Derivs of analytic fcns
Chptr 2: Integrals ...)

Theorem Let A be an open simply connected domain in $\mathbb{R}^2$. Let $[P, Q]$ be a C^1 vector field defined on A. Then there is a function $v \in C^2(A)$ so that

$$v_x = P(x, y), \qquad v_y = Q(x, y)$$

if and only if the curl of the vector field is zero:

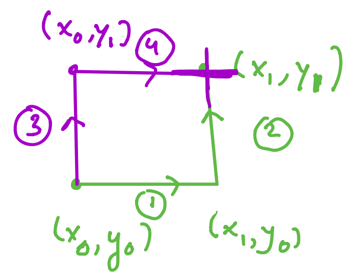
$$P_y = Q_x.$$

This condition is necessary since if v exists then $v_{x\,y} = P_y$ and $v_{y\,x} = Q_x$.

Local proof: (Once we've carefully defined simply-connected domains in Chapter 2, the global theorem in a simply connected region follows from this local version.) Let P, Q be real differentiable, with continuous partials in $B_r((x_0, y_0))$, $r > 0$, and satisfying the "zero curl" condition $P_y = Q_x$. Let $v(x_0, y_0)$ be any chosen constant. Then $\forall$ points $(x_1, y_1) \in B_r((x_0, y_0))$ define $v(x, y)$ in a way which would be consistent with $P = v_x$, $Q = v_y$ if we already knew the function $v(x, y)$. There are two ways to do this using the fundamental theorem of Calculus, and following sides of a rectangle. The curl condition ensures that both routes yield the same value:

$$(1) \qquad v(x_1, y_1) = v(x_0, y_0) + \underbrace{\int_{x_0}^{x_1} \overbrace{P(x, y_0)}^{\frac{\partial v}{\partial x}(x, y_0)}\, dx}_{\text{FTC.}\ v(x_1, y_0)} + \int_{y_0}^{y_1} \overbrace{Q(x_1, y)}^{\frac{\partial v}{\partial y}(x_1, y)}\, dy$$

(1) (2)



$$(2) \qquad v(x_1, y_1) = v(x_0, y_0) + \int_{y_0}^{y_1} \underbrace{Q(x_0, y)}_{\frac{\partial v}{\partial y}(x_0, y)}\, dy + \int_{x_0}^{x_1} \underbrace{P(x, y_1)}_{\frac{\partial v}{\partial x}(x, y_1)}\, dx$$

3 4

The two formulas agree iff the difference of their right hand sides equals zero:

(1) − (2)

$$\int_{x_0}^{x_1} \underbrace{P(x, y_0) - P(x, y_1)}_{1 - 4}\, dx + \int_{y_0}^{y_1} \underbrace{Q(x_1, y) - Q(x_0, y)}_{2 - 3}\, dy = 0$$

iff

FTC   FTC.

$$\int_{x_0}^{x_1} \left(-\int_{y_0}^{y_1} P_y(x, y)\, dy \right) dx + \int_{y_0}^{y_1} \left(\int_{x_0}^{x_1} Q_x(x, y)\, dx \right) dy = 0.$$

$\iint_{\square} \underbrace{-P_y + Q_x}_{0.}\, dx\,dy$

This last equality holds because $-P_y + Q_x = 0$ in the rectangle.

Finally, using (1) and FTC to compute v_{y_1} we see $v_{y_1}(x_1, y_1) = Q(x_1, y_1)$; and using (2) we compute $v_{x_1}(x_1, y_1) = P(x_1, y_1)$ QED.

※ ended here Monday